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ON C-DELTA INTEGRAL OF BANACH SPACE VALUED FUNCTIONS ON TIME SCALES

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ABSTRACT. In this paper we introduce the Banach-valued C-delta integral on time scales and investigate some properties of these integrals.

1. Introduction and preliminaries

The calculus on time scales was introduced for the first time in 1988 by Hilger[2] to unify the theory of difference equations and the theory of differential equations. In 2012, Gwang Sik Eun, Ju Han Yoon, Young Kuk Kim and Byung Moo Kim introduced the C- integral on time scales and inrestigated some properties of the integral. In this paper, we study the Banach-valued C-delta integral on time scales. We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional and F is the indefinite strong C-integral on time scales if and only if the C-variational V_*F is absolutely continuous on time scales.

Throughout this paper, X is a real Banach space with norm $\|\cdot\|$ and its dual X^*. I denote the family of all subintervals of $[a,b]_T$. A time scale T is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in T$ we define the forward jump operator $\sigma(t) = \inf\{s \in T : s > t\}$ where $\inf \phi = \sup T$, while the backward jump operator $\rho(t) = \sup\{s \in T : s < t\}$ where $\sup \phi = \inf T$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ of $t \in T$ is defined by $\mu(t) = \sigma(t) - t$, whlie the

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backward graininess function $\nu(t)$ of $t \in T$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in T$ we denote the closed interval $[a, b]_T = \{t \in T : a \le t \le b\}$. $\delta = (\delta_L, \delta_R)$ is a Δ-gauge on $[a, b]_T$ if $\delta_L(t) > 0$ on $[a, b]_T$, $\delta_R(t) > 0$ on $[a, b]_T$, $\delta_L(a) \ge 0$, $\delta_R(b) \ge 0$ and $\delta_R(t) \ge \mu(t)$ for each $t \in [a, b]_T$.

A collection $P = \{([t_{i-1}, t_i], \xi_i) : 1 \le i \le n\}$ of tagged intervals is

(1) δ-fine McShane partition of $[a, b]_T$ if $[t_{i-1}, t_i]_T \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [a, b]_T$ for each $i = 1, 2, \cdots, n$.

(2) δ-fine C-partition of $[a, b]_T$ if it is a δ-fine McShane partition of $[a, b]_T$ and satisfying the condition

$$\sum_{i=1}^{n} \text{dist}([t_{i-1}, t_i]_T, \xi_i) < \tfrac{1}{\epsilon},$$

where dist $([t_{i-1}, t_i]_T, \xi_i)$=inf$\{|u_i - \xi_i| : u_i \in [t_{i-1}, t_i]_T\}$.

Given a δ-fine C-partition P=$\{([t_{i-1}, t_i]_T, \xi_i)\}_{n=1}^{n}$, we write

$$S(f, P) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$$

for integral sum over P, whenever $f : [a, b]_T \to \mathbb{R}$.

Definition 1.1 ([1]). A function $f : [a, b]_T \longrightarrow \mathbb{R}$ is C- delta integrable on $[a, b]_T$ if there is a number L such that for each $\epsilon > 0$ there exists a Δ-gauge, δ, on $[a, b]_T$ such that

$$|S(f, P) - L| < \epsilon$$

for every δ-fine C- partition $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ of $[a, b]_T$. The number L is called the C-delta integral of f on $[a, b]_T$.

2. On C-delta integral of Banach space valued functions on time scales

In this section, we introduce the C-delta integral of Banach valued functions on time scales and investigate some properties of the integral.

Definition 2.1. A function $f : [a, b]_T \to X$ is C-delta integrable on $[a, b]_T$ if there is a vector $L \in X$ such that for each $\epsilon > 0$ there is a Δ-gauge, δ, on $[a, b]_T$ such that $\|S(f, P) - L\| < \epsilon$ for each $\delta - fine$ C-partition $P = ([t_{i-1}, t_i]_T, \xi_i)_{i=1}^{n}$ of $[a, b]_T$. In this case, L is called the C-integral of f on $[a, b]_T$ and we write $L = \int_a^b f \Delta t$ or $L = (C) \int_a^b f \Delta t$. The function f is C-integrable on a set $E \subset [a, b]_T$ if $f_{\chi E}$ is C-integrable on $[a, b]_T$. We write $\int_E f \Delta t = \int_a^b f_{\chi E} \Delta t$.

By the definition of C-delta integral and similar method of proof of theorem 2.4 in [1], we can easily get the following theorems and Lemma.

THEOREM 2.2. *Let $f : [a, b]_T \to X$ be a function. Then*

1. *if f is C-delta integrable on $[a, b]_T$, then f is C-delta integrable on every subinterval $[c, d]_T$ of $[a, b]_T$.*
2. *if f is C-delta integrable on $[a, c]_T$ and $[c, b]_T$,then f is C-delta integrable on $[a, b]_T$ and $\int_a^b f\Delta t = \int_a^c f\Delta t + \int_c^b f\Delta t$.*

THEOREM 2.3. *let $f, g : [a, b]_T \to X$ be C-delta integrable functions on $[a, b]_T$ and let $\alpha, \beta \in R$. Then $\alpha f + \beta g$ is C-delta integrable function on $[a, b]_T$ and*

$$\int_b^a (\alpha f + \beta g)\Delta t = \alpha \int_a^b f\Delta t + \beta \int_a^b g\Delta t.$$

LEMMA 2.4 (Saks-Henstock Lemma). *Let $f : [a, b]_T \to X$ be C-delta integrable on $[a, b]_T$. Then for each $\epsilon > 0$ there is a Δ- gauge, δ, on $[a, b]_T$. such that*

$$\|S(f, P) - \int_a^b f\Delta t\| < \epsilon$$

for each δ-fine C- partition P of $[a, b]_T$. If $P' = ([t_{i-1}, t_i]_T, \xi_i)_{i=1}^m$ is a δ-fine partial C-partition of $[a, b]_T$, we have

$$\|S(f, P') - \Sigma_{i=1}^m \int_{t_{i-1}}^{t_i} f\Delta t\| \leq \epsilon.$$

THEOREM 2.5. *Let $f : [a, b]_T \to X$ be C-delta integrable function on $[a, b]_T$. Then*

1. *for each $x^* \in X^*$, the function $x^* f$ is C-delta integrable on $[a, b]_T$ and*
$$\int_a^b x^* f\Delta t = x^*\Big(\int_a^b f\Delta t\Big).$$
2. *if Y is a Banach space and $T : X \to Y$ is a continuous linear operator, then Tf is C- delta integrable on $[a, b]_T$ and*
$$\int_a^b Tf\Delta t = T\Big(\int_a^b f\Delta t\Big).$$

DEFINITION 2.6. A finction $f : [a, b]_T \to X$ is strongly C-delta integrable on $[a, b]_T$ if there exist an additive function $F : \mathbf{I} \to X$ such that for each $\epsilon > 0$ there is a Δ-gauge, δ, on $[a, b]_T$ such that

$$\sum_{i=1}^n \|f(\xi_i)(v_i - u_i) - F(u_i, v_i)\| < \epsilon$$

for each δ- fine C- partition $P = ([u_i, v_i]_T, \xi_i)_{i=1}^n$ of $[a, b]_T$. We denote $F(u_i, v_i) = F(v_i) - F(u_i)$.

THEOREM 2.7. *Let X be a Banach space of finite dimension. Then $f : [a, b]_T \to X$ is C- delta integrable on $[a, b]_T$ if and only if f is strongly C-delta integrable on $[a, b]_T$.*

Proof. Let f be strongly C-delta integrable on $[a,b]_T$. By definition 2.6, $f:[a,b]_T \to X$ is C-delta integrable on $[a,b]_T$. Conversely, let $f:[a,b]_T \to X$ be C-delta integrable on $[a,b]_T$. For each $\epsilon > 0$ there is a Δ-gauge, δ, on $[a,b]_T$ such that

$$\|S(f,P) - F(a,b)\| < \epsilon$$

for each δ-fine C-partition $P = ([u,v]_T, \xi)$ of $[a,b]_T$. Let $e_1, e_2, \cdots, e_n$ be a base of X. By the Hahn-Banach Theorem, for each e_i, there is $x_i^* \in X^*$ such that

$$x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ o & \text{if } i \neq j. \end{cases}$$

for $i,j = 1,2,\ldots,n$. Define $g_i = x_i^* f (1 \le i \le n)$, then g_i is C-delta integrable on $[a,b]_T$. For each $\epsilon > 0$ there is a Δ-gauge, δ_i, on $[a,b]_T$ such that

$$|S(g_i, P_i) - \sum \int_u^v g_i \Delta t| \le \tfrac{\epsilon}{2}$$

for each δ- fine C-partition $P_i = ([u,v]_T, \xi)$ of $[a,b]_T$. Since g_i is real valued function. By Saks-Henstock Lemma, we have

$$\sum |g_i(\xi)(u-v) - \int_u^v g_i \Delta t| < \epsilon$$

Also, we have

$$F(u,v) = \int_u^v f \Delta t = \int_u^v (\Sigma_{i=1}^n g_i e_i)\Delta t = \Sigma_{i=1}^n G_i(u,v) e_i$$

where $G_i(u,v) = \int_u^v g_i \Delta t$. Let δ be a positive function on $[a,b]_T$ such that $\delta(x) \le \delta_i(x)$ on $[a,b]_T$ for $i = 1,2,\ldots,n$. For each δ- fine C-partition $P = ([u,v]_T, \xi)$ of $[a,b]_T$, we have

$$\begin{aligned}
\sum \|f(\xi)(v-u) - F(u,v)\| &\le \sum \| \sum_{i=1}^n g_i(\xi) e_i (v-u) - \sum_{i=1}^n G_i(u,v) e_i \| \\
&\le \sum_{i=1}^n \|e_i\| \sum |g_i(\xi)(v-u) - G_i(u,v)| \\
&< \epsilon \sum_{i=1}^n \|e_i\|.
\end{aligned}$$

Thus f is strongly C-delta integrable on $[a,b]_T$. □

3. The C-variational measure and the strong C-integral on time scales

Let $F : [a, b]_T \to X$ and let $E \subset [a, b]_T$ and a $\delta(\xi) : E \to \mathbf{R}^+$ be a positive function. Set

$$V(F, \delta, E) = \sup_{D} \Sigma_i \|F(u_i, v_i)\|,$$

where the supremum is take over all δ - fine partial C-partition $P = ([u_i, v_i]_T, \xi_i)_{i=1}^n$ of $[a, b]_T$ with $\xi_i \in E$. We put

$$V_* F(E) = \inf_{\delta} V(F, \delta, E),$$

where the infimum is take over all function $\delta : E \to R^+$. It is easy to know that the set function $V_* F$ is Borel metric outer measure, known as the C-variational measure generated by F.

Definition 3.1. $V_* F$ is said to be absolutely continuous(AC) on a set $[a, b]_T$ if for each set $N \subset [a, b]$ such that $V_* F(N) = 0$ whenever $\mu(N) = 0$.

Definition 3.2. A function $F : [a, b]_T \to X$ is Δ-differentiable at $t \in [a, b]_T$ if there is a $f(t) \in X$ such that for each $\epsilon > 0$, there exists a a neghborhood $U(t)$ of t such that

$$\|F(\rho(t)) - F(s) - f(\rho(t) - s)\| \leq \epsilon \|\rho(t) - s\|$$

for all $s \in U$ We denote $f(t) = F^{\Delta}(t)$ the Δ- derivative of F at t .

Theorem 3.3. *Let $F : [a, b]_T \to X$ be Δ-differentiable with $f = F^{\Delta}$ a.e. on $[a, b]_T$. then F is the indefinite strong $C-$integral of f if and only if the C-variational measure $V_* F$ is AC.*

Proof. Let $E \subset [a, b]_T$ and $\mu(E) = 0$. Assume $E_n = \{\xi \in E : n - 1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \cdots$. Then we have $E = \cup E_n$ and $\mu(E_n) = 0$, so there are open sets G_n such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n 2^n}$. By The Saks-Henstock Lemma, there exists a positive function δ_0 such that

$$\sum \|S(f, P) - F(u_i, v_i)\| < \epsilon$$

for each δ_0-fine partial C-partition $P = ([u_i, v_i], \xi_i)$ of $[a, b]_T$. Now, for $\xi \in E_n$, take $\delta_n(\xi) > 0$ such that $B(\xi, \delta_n(\xi)) \subset G_n$. and let

$$\delta(\xi) = min\{\delta_0(\xi), \delta_n(\xi)\}.$$

Then for δ-fine partial C-partition $P' = ([u,v],\xi)$ with $\xi \in E$, we have

$$\begin{aligned}\Sigma\|F(u,v)\| &= \Sigma\|F(u,v) - f(\xi)(v-u) + f(\xi)(v-u)\| \\ &\leq \Sigma\|F(u,v) - f(\xi)(v-u)\| + \Sigma\|f(\xi)(v-u)\| \\ &< \epsilon + \Sigma_n \Sigma_{\xi\in E_n}\|f(\xi)(v-u)\| \\ &< \epsilon + \Sigma_n\, n\frac{\epsilon}{n\cdot 2^n} = 2\epsilon\end{aligned}$$

This shows that $V_*F(E) \leq 2\epsilon$. Hence the C-variational measure V_*F is AC. Conversely, there exists a set $E \subset [a,b]_T$ measure zero such that $f(\xi) \neq F^{\Delta}(\xi)$ or $F^{\Delta}(\xi)$ does not exist for $\xi \in E$. Define a function f as follows

$$f(x) = \begin{cases} F^{\Delta}(x) & \text{if } x \in [a,b]_T \cap E^c, \\ \theta & \text{if } x \in E. \end{cases}$$

Then for $\xi \in [a,b]_T \cap E^c$ by the definition of Δ-derivative, for each $\epsilon > 0$ there is a positive function $\delta_1(\xi)$ such that

$$\|f(\xi)(v-u) - F(u,v)\| < \tfrac{\epsilon}{2(b-a)}(dist(\xi,[u,v]) + v - u)$$

for each interval $[u,v]_T \subset (\xi - \delta_1(\xi), \xi + \delta_1(\xi))$. Since V_*F is AC on $[a,b]_T$, then for $\xi \in E$, there is a positive function $\delta_2(\xi)$ such that

$$\sum \|F(u,v)\| < \epsilon$$

for each δ_2 - fine partial C-partition $P_0 = ([u,v],\xi)$ with $\xi \in E$.
Define a positive function $\delta(\xi)$ as follows

$$\delta(\xi) = \begin{cases} \delta_1(\xi) & \text{if } \xi \in [a,b]_T \cap E^c \\ \delta_2(\xi) & \text{if } \xi \in E. \end{cases}$$

Then for each δ-fine C-partition of $[a,b]_T$, we have

$$\begin{aligned}&\sum \|f(\xi)(v-u) - F(u,v)\| \\ &= \sum_{\xi\in E}\|F(u,v) - f(\xi)(v-u)\| + \sum_{\xi\in[a,b]_T\cap E^c}\|F(u,v) - f(\xi)(v-u)\| \\ &\leq \epsilon + \frac{\epsilon}{2(b-a)}\sum_{\xi\in[a,b]\cap E^c}(dist(\xi,[u,v]_T) + v - u) \\ &\leq \epsilon + \frac{\epsilon}{2(b-a)}(2(b-a)) = 2\epsilon.\end{aligned}$$

Hence f is stong C-integrable on $[a,b]_T$ with indefinite strong C-integral F. $\square$

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